L01 Jan6 Topology

Thursday, January 1, 2015 1:31 PM

Qu: In a metric space, how to define an open set G?

① G = a union of balls

② $G = \mathring{G}$, this involves interior points

how is it defined

Qu: Can we define open sets without a metric?

① Ball obviously needs a metric

② Interior point $x \in \mathring{A}$ — needs $B(x,\varepsilon) \subset A$
— $x \in N \subset A$ where N is a nbhd of x i.e. $x \in \mathring{N}$

cyclic argument

Qu: Why is an open set important?

There is a reason about taking limit

However, one single open set alone can have very little information

A system of open sets is important

topology

To define a topology, we start with

open sets (interior) or neighborhoods

Definition Let X be a nonempty set

A set $\mathcal{J} \subset \mathcal{P}(X)$ is a topology for X if

(T1) Any union of sets in $\mathcal{J}$ is still in $\mathcal{J}$

(T2) Any finite intersection of sets in $\mathcal{J}$ is still in $\mathcal{J}$

(T3) $\phi \in \mathcal{J}$, $X \in \mathcal{J}$

Simply put: A topology is closed under arbitary union and finite intersection.

Notation

(T1) $\bigcup \mathcal{A} \in \mathcal{J}$ for all $\mathcal{A} \subset \mathcal{J}$ or

$\bigcup_{\alpha \in I} G_\alpha \in \mathcal{J}$ for all $\{G_\alpha : \alpha \in I\} \subset \mathcal{J}$

(T2) $\bigcap \mathcal{F} \in \mathcal{J}$ for all finite $\mathcal{F} \subset \mathcal{J}$

$G_1 \cap \cdots \cap G_n \in \mathcal{J}$ for all $G_1, \cdots, G_n \in \mathcal{J}$

(T3) $\bigcup \phi = \phi$, $\bigcap \phi = X$

logical consequence of (T1) and (T2)

L01 p3

Friday, January 2, 2015 10:13 AM

Qu. Think of Examples and Non-examples of topologies

Discrete Topology for $X \neq \emptyset$

$\mathcal{J} = \mathcal{P}(X)$. Clearly, T1, T2 are satisfied

Indiscrete Topology for $X \neq \emptyset$

$\mathcal{J} = \{\emptyset, X\}$

Cofinite Topology for $X \neq \emptyset$

$\mathcal{J} = \{ G \subset X : X \setminus G \text{ is finite} \}$

Verify T1 and T2

Think about cases that X is finite / infinite

Standard Euclidean Topology

$X = \mathbb{R}$, $\mathcal{J} = \{(a-\varepsilon, a+\varepsilon) : a \in \mathbb{R}, \varepsilon > 0\} \cup \{\emptyset, \mathbb{R}\}$

Not this one because a union of intervals may not be an interval

Metric Topology Let (X, d) be a metric space

Define $\mathring{A} = \{ x \in A : \exists\, \varepsilon > 0 \ B(x, \varepsilon) \subset A \}$

$\mathcal{J} = \{ G \subset X : G = \mathring{G} \}$

Discrete metric $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

Its metric topology is $\underbrace{\mathcal{P}(X)}_{}$

Why?

How to prove it?

(b) ℓ_p-metric on $\mathbb{R}^n$, $p \geq 1$

$$d(x,y) = \left[\sum_{k=1}^{n} |x_k - y_k|^p \right]^{1/p} \quad \text{or}$$

$$d_\infty(x,y) = \max_{k=1,\cdots,n} |x_k - y_k|$$

All these metrics give the standard topology

Qu: Will $0<p<1$ define a topology?

Qu: Is indiscrete topology a metric topology?

The answer depends on $\#X$

Related Concept Hausdorff or T_2

A topological space $(X, \mathcal{T})$ is Hausdorff if $\forall$ $x,y \in X$ with $x \neq y$ $\exists$ $U,V \in \mathcal{T}$ such that

$x \in U$, $y \in V$, $U \cap V = \emptyset$

Obviously, a metric space is T_2; while an indiscrete space X with $\#X \geq 2$ is not T_2

Qu: Is cofinite topology Hausdorff?

Neighborhoods Given $(X, \mathcal{T})$, $x \in X$

$N \subset X$ is a nbhd of x if

$\exists\, U \in \mathcal{T}$ s.t. $x \in U \subset N$

Prove that the following is satisfied.

For each $x \in X$, there exists

$\mathcal{N}_x \subset \mathcal{P}(X)$, called nbhd system at x

N1 $\forall\, N \in \mathcal{N}_x$, $x \in N$

N2 $\forall\, M, N \in \mathcal{N}_x$, $M \cap N \in \mathcal{N}_x$

N3 If $N \in \mathcal{N}_x$ and $N \subset U$ then $U \in \mathcal{N}_x$

N4 For $N \in \mathcal{N}_x$, denote $\mathring{N} = \{ y \in N : N \in \mathcal{N}_y \}$

then $\mathring{N} \in \mathcal{N}_x$

Note N2 $\Rightarrow \mathcal{N}_x$ is closed under finite $\cap$

N3 $\Rightarrow \mathcal{N}_x$ is closed under arbitrary $\cup$

In the lecture, we will show N4. That involves

1. $x \in \mathring{N}$
2. Can find $U \in \mathcal{T}$ where $x \in U \subset \mathring{N}$

Theorem Suppose a set X has $x \mapsto \mathcal{N}_x \subset \mathcal{P}(X)$ satisfying N1 to N4. Then there is a topology $\mathcal{T}$ for X such that $\mathcal{N}_x$ contains exactly all nbhds of x. need some more thoughts